

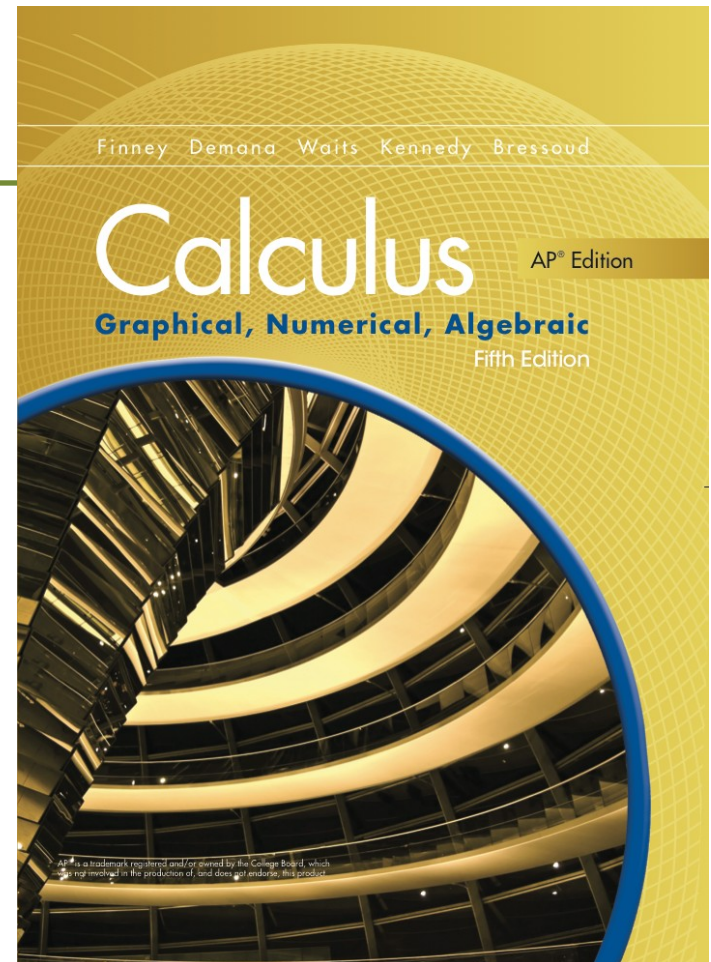
Chapter 10

Infinite Series



Section 10.2

Taylor Series



Quick Review

Find a formula for the n th derivative of the function.

1. e^{3x}

2. 3^x

3. x^n

4. $\ln x$

Find $\frac{dy}{dx}$.

5. $y = \frac{x^n}{n!}$

6. $y = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

7. $y = \frac{(1-x)^n}{n!}$

Quick Review Solutions

Find a formula for the n th derivative of the function.

$$1. e^{3x} \quad 3^n e^{3x}$$

$$2. 3^x \quad 3^x (\ln 3)^n$$

$$3. x^n \quad n!$$

$$4. \ln x \quad (-1)^{n-1} (n-1)! x^{-n}$$

Find $\frac{dy}{dx}$.

$$5. y = \frac{x^n}{n!} \quad \frac{dy}{dx} = \frac{x^{n-1}}{(n-1)!}$$

$$6. y = (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \frac{dy}{dx} = \frac{(-1)^n x^{2n}}{(2n)!}$$

$$7. y = \frac{(1-x)^n}{n!} \quad \frac{dy}{dx} = -\frac{(1-x)^{n-1}}{(n-1)!}$$

What you'll learn about

- Constructing a Maclaurin polynomial using derivatives at 0
- Maclaurin series for $\sin x$ and $\cos x$
- Taylor series and Taylor polynomials
- Observing the convergence of Maclaurin polynomials graphically
- Combining and manipulating Taylor series to form new series
- A table of Maclaurin series and their intervals of convergence

... and why

The partial sums of a Taylor series are polynomials that can be used to approximate the function represented by the series.

Example Constructing a Power Series for $\sin x$

Construct the seventh order Taylor polynomial and the Taylor series for $\sin x$ at $x=0$.

$$\sin(0) = 0$$

$$\sin'(0) = \cos(0) = 1$$

$$\sin''(0) = -\sin(0) = 0$$

$$\sin'''(0) = -\cos(0) = -1$$

$$\sin^{(4)}(0) = \sin(0) = 0$$

$$\sin^{(5)}(0) = \cos(0) = 1$$
 The pattern 0, 1, 0, - 1 repeats forever.

The seventh order Taylor polynomial at $x=0$:

$$\begin{aligned} P_7(x) &= 0 + 1x - 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 - 0x^6 - \frac{1}{7!}x^7 \\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7. \end{aligned}$$

$$\text{The Taylor series: } x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Taylor Series Generated by f at $x = 0$ (Maclaurin Series)

Let f be a function with derivatives of all orders throughout some open interval containing 0. Then the **Taylor series generated by f at $x = 0$** is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is also called the **Maclaurin series generated by f** .

The partial sum $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k$ is the **Taylor polynomial of order n for f at $x = 0$** .

Example Approximating a Function near 0

Find the fourth order Taylor polynomial that approximates $y = \cos 2x$ near $x = 0$.

$$\text{We know that } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\text{Therefore, } \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \dots$$

$$\text{So, } P_4(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} = 1 - 2x^2 + (2/3)x^4.$$

Taylor Series Generated by f at $x=a$

Let f be a function with derivatives of all orders throughout some open interval containing a . Then the **Taylor series generated by f at $x=a$** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The partial sum $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the **Taylor polynomial of order n for f at $x=a$** .

Example A Taylor Series at $x = 1$

Find the Taylor series generated by $f(x) = e^x$ at $x=1$.

Observe that $f(1) = f'(1) = f''(1) = \dots = f^{(n)}(1) = e$

Therefore, the series is $e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \dots + \frac{e}{n!}(x-1)^n + \dots$

$$= \sum_{k=0}^{\infty} \left(\frac{e}{k!} \right) (x-1)^k.$$

Example A Taylor Polynomial for a Polynomial

Find the third order Taylor polynomial for $f(x) = x^3 + 3x^2 - x + 1$ at $x = 0$.

This polynomial is already written in powers of x and is of degree three, so it is its own third order Taylor polynomial at $x = 0$.

Example A Taylor Polynomial for a Polynomial

Find the third order Taylor polynomial for $f(x) = x^3 + 3x^2 - x + 1$ at $x=1$.

$$f(1) = x^3 + 3x^2 - x + 1 \Big|_{x=1} = 4$$

$$f'(1) = 3x^2 + 6x - 1 \Big|_{x=1} = 8$$

$$f''(1) = 6x + 6 \Big|_{x=1} = 12$$

$$f'''(1) = 6$$

$$\text{So, } P_3(x) = 4 + 8(x-1) + \frac{12}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$= (x-1)^3 + 6(x-1)^2 + 8x - 4$$

Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{all real } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (\text{all real } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (\text{all real } x)$$

Maclaurin Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \leq 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (|x| \leq 1)$$